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On the Sturm–Liouville-Type Boundary Value Problem

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The problems of existence, uniqueness and continuous dependence on parameter of solutions of the nonlinear boundary value problem of Sturm–Liouville type for ordinary differential equations are considered. The well-known Banach fixed point theorem is used to establish results. © 1985 Academic Press, Inc.

1. INTRODUCTION

Macki and Waltman [9] have studied the following Sturm–Liouville-type boundary value problem (BVP)

$$x' = f(t, x, y, \lambda), \tag{1}$$

$$y' = g(t, x, y, \lambda), \tag{2}$$

$$A_1 x(a) + A_2\, y(a) = 0, \tag{3}$$

$$B_1 x(b) + B_2\, y(b) = 0, \tag{4}$$

under some suitable conditions. The work of Macki and Waltman [9] examines the existence of eigenvalues of the parameter λ for which there exists a nontrivial solution of BVP (1)–(4). The problems of this type are of great importance from both the theoretical and the applied point of view (see [1, 6, 15]). Our interest in such problems was motivated by their occurrence in various branches of transport theory and in the problems of optimal economic growth; see [7, 8, 12, 13] and the references given therein. Here we assume throughout that $x, y \in C[I, E]$, f, $g \in C[I \times E \times E \times R, E]$, A_i, B_i $(i = 1, 2)$ are constants such that $A_1 \neq 0$, $B_2 \neq 0$, where $I = [a, b]$, E denotes the Banach space with convenient norm $\|\cdot\|$ and R denotes the set of real numbers.

In this paper we present a set of conditions on functions f, g which are sufficient to guarantee that BVP (1)–(4) has a unique solution. We also

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determine conditions for continuous dependence of solutions of BVP (1)–(4) on a parameter λ. Our results are obtained by using the Banach fixed point theorem. The sufficient conditions obtained here are different than those obtained by various authors in [2–5, 7–9, 12–14] and our approach to the problem is different as well.

2. Main Results

In this section we establish our main results on the existence of a unique solution and continuous dependence on parameter of solutions of BVP (1)–(4). For convenience we first list the following hypotheses.

(H_1) For $\lambda \in R$, $(t, x_i, y_i, \lambda) \in I \times E \times E \times R$ $(i = 1, 2)$,

$$\| f(t, x_1, y_1, \lambda) - f(t, x_2, y_2, \lambda) \| \leqslant p(t)[\|x_1 - x_2\| + \|y_1 - y_2\|],$$
$$\| g(t, x_1, y_1, \lambda) - g(t, x_2, y_2, \lambda) \| \leqslant q(t)[\|x_1 - x_2\| + \|y_1 - y_2\|],$$

where $p, q \in C[I, R^+]$ such that

$$\int_a^t p(s) \exp(Ls)\, ds \leqslant \alpha \exp(Lt),$$

$$\int_t^b q(s) \exp(Ls)\, ds \leqslant \beta \exp(Lt),$$

for $t \in I$, in which L is a positive constant and α, β are nonnegative constants such that $0 \leqslant \alpha + \beta < 1$.

(H_2) For every fixed $\lambda \in R$ there exist nonnegative constants N_1, N_2 such that

$$\int_a^t \|f(s, \theta, \theta, \lambda)\|\, ds \leqslant N_1 \exp(Lt),$$

$$\int_t^b \|g(s, \theta, \theta, \lambda)\|\, ds \leqslant N_2 \exp(Lt),$$

for $t \in I$, where θ is the zero element in E.

(H_3) There exist nonnegative constants M_1, M_2 and functions $p_0, q_0 \in C[I, R^+]$ such that for $(t, x, y, \lambda_i) \in I \times E \times E \times R$ $(i = 1, 2)$,

$$\|f(t, x, y, \lambda_1) - f(t, x, y, \lambda_2)\| \leqslant p_0(t)\, |\lambda_1 - \lambda_2|,$$
$$\|g(t, x, y, \lambda_1) - g(t, x, y, \lambda_2)\| \leqslant q_0(t)\, |\lambda_1 - \lambda_2|,$$

and

$$\int_a^t p_0(s)\,ds \leqslant M_1 \exp(Lt),$$

$$\int_t^b q_0(s)\,ds \leqslant M_2 \exp(Lt),$$

for $t \in I$.

Let $B = E \times E$ be the product space equipped with the norm

$$\|z(t)\|_B = \|z_1(t)\| + \|z_2(t)\|, \qquad z = (z_1, z_2) \in B,\ t \in I.$$

Let S be a space of those functions $z = (z_1, z_2) \in B$, which are continuous for $t \in I$ and fulfill the condition

$$\|z(t)\|_B = O(\exp(Lt)), \qquad t \in I. \tag{5}$$

In the space S we define the norm (see [10])

$$\|z\|_1 = \sup_{t \in I}[\|z(t)\|_B \exp(-Lt)]. \tag{6}$$

It is easy to see that S with the norm defined in (6) is a Banach space.

We note that the condition (5) implies that there exists a constant $M \geqslant 0$ such that $\|z(t)\|_B \leqslant M \exp(Lt)$, $t \in I$. Using this fact in (6) we observe that

$$\|z\|_1 \leqslant M. \tag{7}$$

We now establish the following existence–uniqueness theorem for BVP (1)–(4).

THEOREM 1. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then for every* $\lambda \in R$ *there exists a unique solution* $(x, y) \in S$ *of* BVP (1)–(4).

Proof. Let $\lambda \in R$ be fixed. For $z = (x, y) \in S$ we define the operator $Tz = (T_1 x, T_2 y)$, where the operators T_1, T_2: $E \to E$ are defined by

$$T_1 x(t) = c_1 + \int_a^t f(s, x(s), y(s), \lambda)\,ds, \tag{8}$$

$$T_2 y(t) = c_2 - \int_t^b g(s, x(s), y(s), \lambda)\,ds, \tag{9}$$

where $c_1 = -(A_2/A_1)\, y(a)$ and $c_2 = -(B_1/B_2)\, x(b)$. Clearly the solution of BVP (1)–(4) is a fixed point of the operator equation $Tz = z$. We first prove that T maps S into itself. Evidently T is continuous and $Tz \in E \times E$ for $t \in I$.

We verify that (5) is fulfilled. From (8) and (9) and using (H_1) and (H_2) and (7) we obtain

$$\begin{aligned}
\|T_1 x(t)\| &\leqslant \|c_1\| + \int_a^t \|f(s, x(s), y(s), \lambda) - f(s, \theta, \theta, \lambda)\|\, ds \\
&\quad + \int_a^t \|f(s, \theta, \theta, \lambda)\|\, ds \\
&\leqslant \|c_1\| + \int_a^t p(s)\, \|z(s)\|_B \exp(-Ls) \exp(Ls) + N_1 \exp(Lt) \\
&\leqslant \|c_1\| + \|z\|_1 \alpha \exp(Lt) + N_1 \exp(Lt) \\
&\leqslant [\|c_1\| + M\alpha + N_1] \exp(Lt), \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
\|T_2 y(t)\| &\leqslant \|c_2\| + \int_t^b \|g(s, x(s), y(s), \lambda) - g(s, \theta, \theta, \lambda)\|\, ds \\
&\quad + \int_t^b \|g(s, \theta, \theta, \lambda)\|\, ds \\
&\leqslant \|c_2\| + \int_t^b q(s)\, \|z(s)\|_B \exp(-Ls) \exp(Ls)\, ds + N_2 \exp(Lt) \\
&\leqslant \|c_2\| + \|z\|_1\, \beta \exp(Lt) + N_2 \exp(Lt) \\
&\leqslant [\|c_2\| + M\beta + N_2] \exp(Lt). \qquad (11)
\end{aligned}$$

From (10) and (11) we have

$$\|Tz\|_1 \leqslant [\|c_1\| + \|c_2\| + M(\alpha + \beta) + N_1 + N_2].$$

This proves that T maps S into itself.

Now we verify that the operator T is a contraction map. Let $z = (x, y)$, $\bar{z} = (\bar{x}, \bar{y}) \in S$ be such that $Tz = (T_1 x, T_2 y)$, $T\bar{z} = (T_1 \bar{x}, T_2 \bar{y})$. From hypothesis (H_1) we have

$$\begin{aligned}
\|T_1 x(t) - T_1 \bar{x}(t)\| &\leqslant \int_a^t \|f(s, x(s), y(s), \lambda) - f(s, \bar{x}(s), \bar{y}(s), \lambda)\|\, ds \\
&\leqslant \int_a^t p(s)\, \|z(s) - \bar{z}(s)\|_B \exp(-Ls) \exp(Ls)\, ds \\
&\leqslant \alpha\, \|z - \bar{z}\|_1 \exp(Lt), \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}\|T_2\,y(t)-T_2\,\bar{y}(t)\| &\leqslant \int_t^b \|g(s,x(s),y(s),\lambda)-g(s,\bar{x}(s),\bar{y}(s),\lambda)\|\,ds\\ &\leqslant \int_t^b q(s)\,\|z(s)-\bar{z}(s)\|_B \exp(-Ls)\exp(Ls)\,ds\\ &\leqslant \beta\,\|z-\bar{z}\|_1 \exp(Lt). \end{aligned} \tag{13}$$

From (12) and (13) we have

$$\|Tz-T\bar{z}\|_1 \leqslant (\alpha+\beta)\,\|z-\bar{z}\|_1. \tag{14}$$

Since $\alpha+\beta<1$, it follows from the Banach fixed point theorem that T has a unique fixed point in S. The fixed point of T is however a solution of BVP (1)–(4). This completes the proof of the theorem.

The following theorem determine conditions for continuous dependence of solutions of BVP (1)–(4) on a parameter λ. The idea of the proof is based on the result given by the present author in [10, Theorem 2] (see also [11]).

THEOREM 2. *Let* (H_1), (H_2) *and* (H_3) *hold. Then the soluion* $(x(t,\lambda), y(t,\lambda))$ *of* BVP (1)–(4) *belonging to* S *is continuous with respect to the variables* (t,λ) *in* $I\times R$.

Proof. For $z=(x,y)\in S$ we define the operator $T_\lambda z=(T_{1\lambda}x, T_{2\lambda}\,y)$, where $T_{1\lambda}x$ and $T_{2\lambda}\,y$ are defined by the right sides of the equations (8) and (9), respectively. From (14) we have

$$\|T_\lambda z-T_\lambda\bar{z}\|_1 \leqslant (\alpha+\beta)\,\|z-\bar{z}\|_1. \tag{15}$$

Next from hypothesis (H_3) we obtain for $\lambda, \lambda_0\in R$,

$$\begin{aligned}\|T_{1\lambda}x(t)-T_{1\lambda_0}x(t)\| &\leqslant \int_a^t \|f(s,x(s),y(s),\lambda)-f(s,x(s),y(s),\lambda_0)\|\,ds\\ &\leqslant \int_a^t p_0(s)\,|\lambda-\lambda_0|\,ds\\ &\leqslant M_1\,|\lambda-\lambda_0|\exp(Lt), \end{aligned} \tag{16}$$

and

$$\begin{aligned}\|T_{2\lambda}\, y(t) - T_{2\lambda_0}\, y(t)\| &\leqslant \int_t^b \|g(s, x(s), y(s), \lambda) - g(s, x(s), y(s), \lambda_0)\|\, ds \\ &\leqslant \int_t^b q_0(s)\, |\lambda - \lambda_0|\, ds \\ &\leqslant M_2\, |\lambda - \lambda_0| \exp(Lt). \qquad (17)\end{aligned}$$

From (16) and (17) we have

$$\|T_\lambda z - T_{\lambda_0} z\|_1 \leqslant (M_1 + M_2)\, |\lambda - \lambda_0|. \qquad (18)$$

From Theorem 1 there exists a unique function $z(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ such that $T_\lambda z(t, \lambda) = (T_{1\lambda} x(t, \lambda),\ T_{2\lambda}\, y(t, \lambda)) = z(t, \lambda)$ and $T_{\lambda_0} z(t, \lambda_0) = (T_{1\lambda_0} x(t, \lambda_0),\ T_{2\lambda_0}\, y(t, \lambda_0)) = z(t, \lambda_0)$ for $t \in I$ and $\lambda, \lambda_0 \in R$. From (15) and (18) we have

$$\begin{aligned}\|z(t, \lambda) - z(t, \lambda_0)\|_1 &\leqslant \|T_\lambda z(t, \lambda) - T_\lambda z(t, \lambda_0)\|_1 \\ &\quad + \|T_\lambda z(t, \lambda_0) - T_{\lambda_0} z(t, \lambda_0)\|_1 \\ &\leqslant (\alpha + \beta)\, \|z(t, \lambda) - z(t, \lambda_0)\|_1 + (M_1 + M_2)\, |\lambda - \lambda_0|,\end{aligned}$$

and hence

$$\|z(t, \lambda) - z(t, \lambda_0)\|_1 \leqslant (M_1 + M_2)[1 - (\alpha + \beta)]^{-1}\, |\lambda - \lambda_0|.$$

This shows that the function $z(t, \lambda)$ is continuous with respect to the variable λ in R uniformly with respect to the variable t in I and consequently $z(t, \lambda)$ is also continuous with respect to the two variables (t, λ) in $I \times R$. The proof of the theorem is complete.

We note that BVP (1)–(4) is of more general type and contains as a special case the boundary value problems for second order equations studied by Baily [1]. In [7, 8, 12, 13] the authors have studied BVP (1)–(4) when $\lambda = 0$ by using different techniques. For earlier results on the same type of problems, see [2–5, 14]. An important feature of our approach here is that it is elementary and provides a uniform treatment for problems of this type.

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